Exercise 9

Show that to represent each point in $\mathbb{R}^3$ by spherical coordinates, it is necessary to take only values of θ between 0 and 2π, values of ϕ between 0 and π, and values of $\rho \geq 0$. Are coordinates unique if we allow $\rho \leq 0$?

Solution

By definition, the spherical coordinates of a point (x, y, z) in space are

$$x = \rho \sin\phi \cos\theta \tag{1}$$
$$y = \rho \sin\phi \sin\theta \tag{2}$$
$$z = \rho \cos\phi. \tag{3}$$

Square both sides of each equation and add the respective sides.

$$\begin{aligned} x^2 + y^2 + z^2 &= (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 + (\rho \cos\phi)^2 \\ &= \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \cos^2\phi \\ &= \rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2 \cos^2\phi \\ &= \rho^2 \sin^2\phi + \rho^2 \cos^2\phi \\ &= \rho^2(\sin^2\phi + \cos^2\phi) \\ &= \rho^2 \end{aligned}$$

Take the square root of both sides to get ρ.

$$\rho = \pm\sqrt{x^2 + y^2 + z^2}$$

Either sign can be taken because the other terms on the right sides of equations (1), (2), and (3) take care of the sign as x, y, and z go from $-\infty$ to ∞. By convention, the positive sign is chosen.

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

As $-\infty < x, y, z < \infty$, the square root function yields positive values: $\rho \geq 0$. θ is only present in equations (1) and (2) in the $\cos\theta$ and $\sin\theta$ terms. Both of these functions are 2π-periodic, so an interval 2π units long for θ is needed. To be consistent with polar coordinates, it's chosen to be $0 \leq \theta < 2\pi$. Solve equation (3) for ϕ.

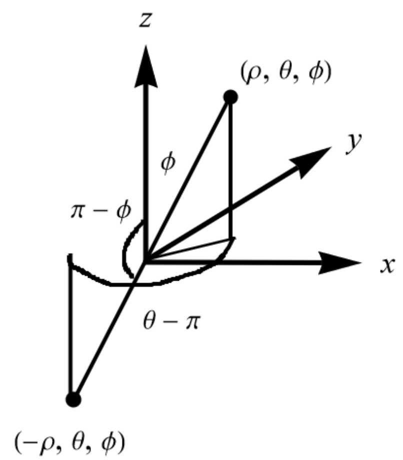
$$\cos\phi = \frac{z}{\rho}$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$$

$$\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

As $-\infty < x, y, z < \infty$, the values that the inverse cosine yields are $0 \leq \phi \leq \pi$. This justifies the comparison with polar coordinates because $\rho \sin\phi$ is nonnegative and can be thought of as r.

It's pointless to consider both negative and positive values of ρ because then any point in space can be represented in more than one way. Changing ρ to $-\rho$ reflects the point through the origin as shown below. Changing θ to $\theta - \pi$ and ϕ to $\pi - \phi$ brings it back to the original point.



A point represented by (ρ, θ, ϕ) can be represented by $(-\rho, \theta - \pi, \pi - \phi)$ because

$$\begin{aligned}
x &= \rho \sin\phi \cos\theta = -\rho \sin(\pi - \phi) \cos(\theta - \pi) = -\rho(\sin\phi)(-\cos\theta) = \rho \sin\phi \cos\theta = x \\
y &= \rho \sin\phi \sin\theta = -\rho \sin(\pi - \phi) \sin(\theta - \pi) = -\rho(\sin\phi)(-\sin\theta) = \rho \sin\phi \sin\theta = y \\
z &= \rho \cos\phi = -\rho \cos(\pi - \phi) = -\rho(-\cos\phi) = \rho \cos\phi = z.
\end{aligned}$$